

# Representation Theorems of $\mathbb{R}$ -trees

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**Abstract.** In this paper, we provide a new representation of an  $\mathbb{R}$ -tree by using a set of graphs. We have captured the four-point condition from these graphs and identified the radial metric and river metric by some particular graphical representations. In stochastic analysis, these representation theorems are of particular interest in identifying Brownian motions indexed by  $\mathbb{R}$ -trees.

## 1 Introduction

One of the central object of probability is Brownian motion (Bm), which is the microscopic picture emerging from a particle moving in  $n$ -dimensional space and of course the nature of Brownian paths is of special interest. In this paper, we study the features of Brownian motion indexed by an  $\mathbb{R}$ -tree. An  $\mathbb{R}$ -tree is a 0-hyperbolic metric space with desirable properties. Note that, J. Istaş in [8] proved that the fractional Brownian motion can be well defined on a hyperbolic space when  $0 < H \leq \frac{1}{2}$ . Furthermore, in [2] the authors use Dirichlet form methods to construct Brownian motion on any given locally compact  $\mathbb{R}$ -tree, additionally in [7], representation of a Gaussian field via a set of independent increments were discussed. In this paper we focus on “radial” and “river” metric and clarify the relationship between metric trees generated by these metrics and a particular metric ray denoted by  $\mathcal{C}_d(A, B)$ . Our investigation is motivated by the questions: under what conditions on the set  $\{\mathcal{C}_d(A, B)\}_{A, B \in M}$  does  $(M, d)$  become an  $\mathbb{R}$ -tree? and when can an  $\mathbb{R}$ -tree be identified by the sets  $\{\mathcal{C}_d(A, B)\}_{A, B \in M}$ ? It is our hope that this work could lead to the interest of applying those results to random fields indexed by metric spaces.

The study of injective envelopes of metric spaces, also known as  $\mathbb{R}$ -trees (metric trees or  $T$ -theory) began with J. Tits in [12] in 1977 and since then, applications have been found within many fields of mathematics. For a complete discussion of these spaces and their relation to global metric spaces of nonpositive curvature we refer to [4]. Applications of metric trees in biology and medicine stems from the construction of phylogenetic trees [11]. Concepts of “string matching” in computer science are closely related with the structure of metric trees [3].  $\mathbb{R}$ -trees are a generalization of an ordinary tree which allows for different weights on edges. In order to define an  $\mathbb{R}$ -tree, we first introduce the notion of metric segment. Let  $(M, d)$  be a metric space. For any  $A, B \in M$ , the *metric segment*  $[A, B]$  is defined by

$$[A, B] = \{X \in M : d(A, X) + d(X, B) = d(A, B) < +\infty\}.$$

Notice that by this definition,  $[A, B] \neq \emptyset$  if and only if  $A, B$  are connected in  $(M, d)$ .

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**Definition 1.1** (see [10]) *An  $\mathbb{R}$ -tree is a nonempty metric space  $(M, d)$  satisfying:*

- (a) *Any two points of  $A, B \in M$  are joined by a unique metric segment  $[A, B]$ .*
- (b) *If  $A, B, C \in M$ , then*

$$[A, B] \cap [A, C] = [A, O] \text{ for some } O \in M.$$

- (c) *If  $A, B, C \in M$  and  $[A, B] \cap [B, C] = \{B\}$ , then*

$$[A, B] \cup [B, C] = [A, C].$$

Through out this paper we only consider the class of metric spaces satisfying (a) in Definition 1.1 above. We call this metric space uniquely geodesic metric space. In the following we characterize an  $\mathbb{R}$ -tree by the theorem below (given in [5]):

**Theorem 1.1** *A uniquely geodesic metric space  $(M, d)$  is an  $\mathbb{R}$ -tree if and only if it is connected, contains no triangles and satisfies the four-point condition (4PC).*

Note that, we say a metric  $d(\cdot, \cdot)$  satisfies the four-point condition (4PC) if, for any  $A, B, C, D$  in  $M$  the following inequality holds:

$$d(A, B) + d(C, D) \leq \max\{d(A, C) + d(B, D), d(A, D) + d(B, C)\}.$$

The four-point condition is stronger than the triangle inequality (take  $C = D$ ), but it should not be confused with the ultrametric definition. An ultrametric satisfies the condition  $d(A, B) \leq \max\{d(A, C), d(B, C)\}$ , and this is stronger than the four-point condition. Moreover, we say  $A, B, C$  form a triangle if all the triangle inequalities involving  $A, B, C$  are strict and  $[X, Y] \cap [Y, Z] = \{Y\}$  for any  $\{X, Y, Z\} = \{A, B, C\}$ .  $d(\cdot, \cdot)$  is said to be a tree metric if it satisfies the (4PC). Given a metric space  $(M, d)$ , we would capture the tree metric properties of  $(M, d)$  by introducing the following sets  $\{\mathcal{C}_d(A, B)\}_{A, B \in M}$ .

**Definition 1.2** *For any  $A, B \in M$ , we define*

$$\mathcal{C}_d(A, B) = \{X \in M : d(X, A) = d(X, B) + d(A, B) < +\infty\}.$$

Note that two points  $A, B \in M$  are connected if and only if  $\mathcal{C}_d(A, B) \neq \emptyset$ .

We remark that a Brownian motion is uniquely determined by independent increments and furthermore, since the set  $\mathcal{C}_d(P_1, P_2)$  is also defined as:

$$\mathcal{C}_d(P_1, P_2) = \{X \in M : B(X) - B(P_2) \text{ is independent of } B(P_2) - B(P_1)\},$$

It is of interest to ask the following questions:

Question 1: Under what conditions on the set  $\{\mathcal{C}_d(A, B)\}_{A, B \in M}$  does  $(M, d)$  become an  $\mathbb{R}$ -tree?

Question 2: When can an  $\mathbb{R}$ -tree be identified by the set  $\{\mathcal{C}_d(A, B)\}_{A, B \in M}$ ?

In this paper we give complete solution to Question 1 (see Section 2 below), namely, we provide a sufficient and necessary condition on  $\{\mathcal{C}_d(A, B)\}_{A, B \in M}$  such that  $(M, d)$  is an  $\mathbb{R}$ -tree. In Section 3.1, we study Question 2 by considering radial metric and river metric. We show that the answer to Question 2 is positive when  $M = \mathbb{R}^n$  and

$$d(A, B) = g_k(|A - B|) \text{ for } A, B \in \Pi_k,$$

where  $|\cdot|$  is the Euclidean norm,  $(\Pi_k)_{k=1, \dots, N}$  is some partition of  $\mathbb{R}^n$  and  $g_k : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a continuous function.

## 2 Results : An Equivalence of $\mathbb{R}$ -tree Properties

We start by introducing the following conditions that will be used in the proof of the Theorem 2.1:

*Condition (A): For any 3 distinct points  $A, B, C \in M$ , there exists unique  $O \in M$  such that*

$$\{X, Y\} \subset \mathcal{C}_d(Z, O) \text{ for any distinct } X, Y, Z \in \{A, B, C\}.$$

*Condition (B): For any distinct  $A, B, C \in M$ , there exists  $O \in M$  such that*

$$[A, B] \cap [B, C] \cap [A, C] = \{O\}.$$

Note that if the cardinality  $|M| = 1$  or  $2$ , then  $(M, d)$  is obviously an  $\mathbb{R}$ -tree, since any 2 points are joined by a unique geodesic. When  $|M| \geq 3$ , Condition (A) guarantees that  $(M, d)$  contains no circuit. The following will be used in the proof of the Theorem 2.1:

**Lemma 2.1** *Condition (A) is equivalent to Condition (B).*

**Proof.** We only consider the case where  $M$  contains at least 3 distinct points. Let's pick 3 distinct points  $A, B, C \in M$ . Then by observing that for any distinct  $X, Y \in \{A, B, C\}$ ,

$$X \in \mathcal{C}_d(Y, O) \text{ is equivalent to } O \in [X, Y].$$

Thus Lemma 2.1 holds.  $\square$

**Theorem 2.1** *A uniquely geodesic metric space  $(M, d)$  is an  $\mathbb{R}$ -tree if and only if Condition (A) holds.*

**Proof** By Lemma 2.1, it is sufficient to prove Theorem 2.1 holds under Condition (B). First we show that if  $(M, d)$  is an  $\mathbb{R}$ -tree, then Condition (B) is satisfied, and then show that if Condition (B) holds, then  $(M, d)$  is an  $\mathbb{R}$ -tree. Suppose  $(M, d)$  is an  $\mathbb{R}$ -tree, since  $(M, d)$  is connected, then  $[A, B] \neq \emptyset$  for all  $A, B \in M$ . For any 3 points  $A, B, C \in M$  we have:

- if  $[A, B] \cap [B, C] = \{B\}$ , then by (c) in Definition 1.1,

$$\{B\} = [A, B] \cap [B, C] \subset [A, B] \cup [B, C] = [A, C].$$

This yields

$$[A, B] \cap [B, C] \cap [A, C] = \{B\} \cap [A, C] = \{B\}.$$

- If there exists  $O \in M$ ,  $O \neq B$  such that  $[A, B] \cap [B, C] = [B, O]$ , then  $O \in [A, B] \cap [B, C] \cap [A, C]$ . Thus, condition (B) is verified.

Next assume that Condition (B) holds. By taking any  $A \neq B = C$ , one easily shows that  $[A, B] \neq \emptyset$ , thus  $(M, d)$  is connected. The fact that  $[A, B] \cap [B, C] \cap [A, C] \neq \emptyset$  leads to the fact that there is no triangles in  $(M, d)$ . Then it is sufficient to prove that  $d(\cdot, \cdot)$  satisfies the (4PC). Let us pick 4 distinct points  $A, B, C, D$  from  $M$ . Under Condition (B), we have two possibilities to the positions of  $A, B, C, D$ , namely:

1.  $A, B, C, D$  are in the same metric segment.
2. Case 1 above does not hold.

Case 1 is equivalent to

$$\begin{cases} d(W, X) = d(W, Z) + d(X, Z); \\ d(W, Y) = d(W, Z) + d(Y, Z). \end{cases}$$

This easily leads to the (4PC). For the second case, for any  $\{X, Y, Z, W\} = \{A, B, C, D\}$ , if  $\mathcal{C}_d(X, Z) = \mathcal{C}_d(Y, Z)$ , then one necessarily has  $W \notin \mathcal{C}_d(X, Z) = \mathcal{C}_d(Y, Z)$ . This is equivalent to

$$\begin{cases} d(W, X) < d(W, Z) + d(X, Z); \\ d(W, Y) < d(W, Z) + d(Y, Z). \end{cases}$$

This easily leads to the (4PC). It is easy to check that in each case, the (4PC) is satisfied.  $\square$

## 2.1 Characterization of $\mathcal{C}_d(P_1, P_2)$ via Radial Metric

We define an  $\mathbb{R}$ -tree  $(\mathbb{R}^n, d_1)$  ( $n \geq 1$ ) with root 0 and the metric

$$d_1(A, B) = \begin{cases} |A - B| & \text{if } A = aB \text{ for some } a \in \mathbb{R}; \\ |A| + |B| & \text{otherwise.} \end{cases}$$

We explicitly represent the set  $\mathcal{C}_{d_1}(P_1, P_2)$  for all  $P_1, P_2 \in \mathbb{R}^n$  in the following proposition.

**Proposition 2.1** *For any  $P_1, P_2 \in (\mathbb{R}^n, d_1)$ ,*

$$\mathcal{C}_{d_1}(P_1, P_2) = \begin{cases} [P_2, +\infty)_{\overrightarrow{0P_2}} & \text{if } P_2 \notin [0, P_1]_{\overrightarrow{0P_1}}; \\ \mathbb{R}^n \setminus (P_2, +\infty)_{\overrightarrow{0P_1}} & \text{if } P_2 \in [0, P_1]_{\overrightarrow{0P_1}}; \\ \mathbb{R}^n & \text{if } P_1 = P_2, \end{cases} \quad (2.1)$$

where for any  $A, B \in \mathbb{R}^n$ ,  $[A, B]_{\overrightarrow{AB}}$  denotes the segment  $\{(1-a)A + aB; a \in [0, 1]\}$  and  $(A, +\infty)_{\overrightarrow{0B}}$  denotes  $\{aA + bB; a > 1, b > 0\}$  under Euclidean distance. These notations shouldn't be confused with the metric segments  $[A, B]$  of a metric space.

**Proof.** Since it is always true that  $\mathcal{C}_{d_1}(P_1, P_2) = \mathbb{R}^n$  for  $P_1 = P_2$ , then we only consider the case when  $P_1 \neq P_2$ . There are 3 different situations to the positions of  $P_1, P_2$ : (1)  $P_1, P_2$  are on the same ray (which means,  $P_2 = aP_1$  for some  $a \in \mathbb{R}$ ) and  $0 \leq |P_1| < |P_2|$ ; (2)  $P_1, P_2$  are on the same ray and  $0 \leq |P_2| < |P_1|$ ; (3)  $P_1, P_2$  are on different rays.

Case (1):  $P_1, P_2$  are on the same ray and  $0 \leq |P_1| < |P_2|$ .

In this case one necessarily has

$$d_1(A, P_1) = d_1(A, P_2) + |P_1 - P_2|. \quad (2.2)$$

Case (1.1): If  $A$  is on a different ray as  $P_2$ , then (2.2) becomes

$$|A| + |P_1| = |A| + |P_2| + |P_1 - P_2|.$$

This together with the fact that  $P_1 \neq P_2$  implies

$$|P_1| = |P_2| + |P_1 - P_2| > |P_2|.$$

This is impossible, because it is assumed that  $|P_1| < |P_2|$ .

Case (1.2): Suppose  $A$  is on the same ray as  $P_2$ . Now (2.2) is equivalent to

$$|A - P_1| = |A - P_2| + |P_1 - P_2|.$$

The solution space for  $A$  is then the segment  $[P_2, +\infty)_{\overrightarrow{OP_2}}$  under Euclidean distance.

One concludes that in Case (1),

$$\mathcal{C}_{d_1}(P_1, P_2) = [P_2, +\infty)_{\overrightarrow{OP_2}}. \quad (2.3)$$

Case (2):  $P_1, P_2$  are on the same ray and  $|P_1| > |P_2| \geq 0$ . Note that (2.2) still holds.

Case (2.1): Suppose that  $A$  is on a different ray as  $P_1$ . (2.2) is then equivalent to

$$|P_1| = |P_2| + |P_1 - P_2|.$$

The above equation always holds true. Therefore any  $A$  on a different ray as  $P_1$  belongs to  $\mathcal{C}_{d_1}(P_1, P_2)$ . Case (2.2):  $A$  is on the same ray as  $P_1$ . Equation (2.2) then becomes

$$|A - P_1| = |A - P_2| + |P_1 - P_2|,$$

and its solution space is segment  $[0, P_2]_{\overrightarrow{OP_2}}$  under Euclidean distance.

Combining Case (2.1) and Case (2.2), one obtains, in Case (2),

$$\mathcal{C}_{d_1}(P_1, P_2) = \mathbb{R}^n \setminus (P_2, +\infty)_{\overrightarrow{OP_1}}. \quad (2.4)$$

Case (3):  $P_1, P_2$  are on different rays with  $P_1, P_2 \neq 0$ .

Case (3.1):  $A$  is on the same ray as  $P_1$ .

In this case we have

$$|A - P_1| = |A| + |P_2| + |P_1| + |P_2|.$$

By the triangle inequality,

$$|A| + |P_1| + 2|P_2| = |A - P_1| \leq |A| + |P_1|.$$

This yields the absurd statement  $P_2 = 0$ !

Case (3.2):  $A$  is on the same ray as  $P_2$ .

We have

$$|A| + |P_1| = |A - P_2| + |P_1| + |P_2|.$$

This leads to  $A \in [P_2, +\infty)_{\overrightarrow{OP_2}}$ .

Case (3.3):  $A$  is on a different ray as  $P_1, P_2$ .

We have

$$|A| + |P_1| = |A| + |P_2| + |P_1| + |P_2|,$$

which again implies  $P_2 = 0$ . Contradiction!

We conclude that in Case (3),

$$\mathcal{C}_{d_1}(P_1, P_2) = [P_2, +\infty)_{\overrightarrow{OP_2}}. \quad (2.5)$$

Finally, by combining (2.3), (2.4) and (2.5), one proves Proposition 2.1 (see Figure 1, Figure 2).  $\square$

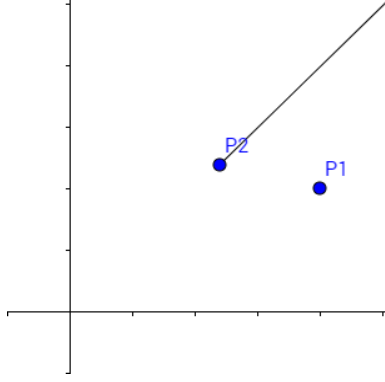


Figure 1: The thick line represents the set of  $\mathcal{C}_{d_1}(P_1, P_2)$  when  $P_2$  is not in the segment  $[0, P_1]$ .

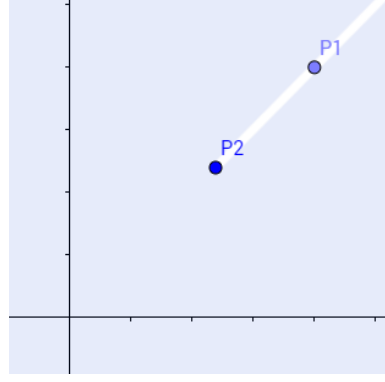


Figure 2: The shaded region represents  $\mathcal{C}_{d_1}(P_1, P_2)$  when  $P_2$  is in the segment  $[0, P_1]$ .

Now we would show the inverse of Proposition 2.1, namely whether or not we can recognize  $(\mathbb{R}^n, d)$  as an  $\mathbb{R}$ -tree using  $\mathcal{C}_d(P_1, P_2)$ . For that purpose, we first show the following statement.

**Proposition 2.2** *Let  $(\mathbb{R}^n, d)$  be a metric space. If (2.1) holds for any  $P_1, P_2 \in (\mathbb{R}^n, d)$ , then  $(\mathbb{R}^n, d)$  is an  $\mathbb{R}$ -tree.*

**Proof.** By Theorem 2.1, we only need to show *Condition (B)* holds. Let us arbitrarily pick 3 different points  $A, B, C \in \mathbb{R}^n$ . If  $A, B, C$  are in the same segment, saying,  $A \in \mathcal{C}_d(B, C)$ , then  $C \in [A, B] \cap [B, C] \cap [A, C]$  and *Condition (B)* is satisfied. If  $A, B, C$  are not in the same segment, i.e.,  $X \notin \mathcal{C}_d(Y, Z)$  for any  $\{X, Y, Z\} = \{A, B, C\}$ , then we see from the definition of  $\mathcal{C}_d(P_1, P_2)$  that

$$X \in \mathcal{C}_d(Y, 0) \text{ for any distinct } X, Y \in \{A, B, C\},$$

which is equivalent to  $0 \in [A, B] \cap [B, C] \cap [A, C]$ . Hence *Condition (B)* is satisfied.  $\square$

## 2.2 Characterization of $\mathcal{C}_d(P_1, P_2)$ via River Metric

For  $A \in \mathbb{R}^2$ , we denote by  $A = (A^{(1)}, A^{(2)})$ . We define the river metric space  $(\mathbb{R}^2, d_2)$  by taking

$$d_2(A, B) = \begin{cases} |A^{(2)} - B^{(2)}| & \text{for } A^{(1)} = B^{(1)}; \\ |A^{(2)}| + |A^{(1)} - B^{(1)}| + |B^{(2)}| & \text{for } A^{(1)} \neq B^{(1)}. \end{cases}$$

From now on we say that  $A, B$  are on the same ray in  $(\mathbb{R}^2, d_2)$  if and only if  $A, B$  are on a vertical Euclidean line:  $A^{(1)} = B^{(1)}$ .

**Proposition 2.3** Let  $(\mathbb{R}^2, d_2)$  be a river metric space. For  $P \in \mathbb{R}^2$ , denote by  $P^* = (P^{(1)}, 0)$  the projection of  $P$  to the horizontal axis. Then for any  $P_1, P_2 \in \mathbb{R}^2$ , we have

$$\mathcal{C}_{d_2}(P_1, P_2) = \begin{cases} \mathbb{R}^2 \setminus (P_2, \infty)_{\overrightarrow{P_1^* P_1}} & \text{if } P_2 \in [P_1^*, P_1); \\ [P_2, \infty)_{\overrightarrow{P_2^* P_2}} & \text{if } P_2 \notin [P_1^*, P_1) \text{ and } P_2^{(2)} \neq 0; \\ [P_2^{(1)}, \infty)_{\overrightarrow{P_1^{(1)} P_2^{(1)}}} \times \mathbb{R} & \text{if } P_1^{(1)} \neq P_2^{(1)}, P_2^{(2)} = 0; \\ \mathbb{R}^2 & \text{if } P_1 = P_2. \end{cases} \quad (2.6)$$

**Proof.** It is obvious that  $\mathcal{C}_{d_2}(P_1, P_2) = \mathbb{R}^2$  when  $P_1 = P_2$ . For  $P_1 \neq P_2$ , we mainly consider 2 cases: (1)  $P_1, P_2$  are on the same ray ( $P_1^{(1)} = P_2^{(1)}$ ); (2)  $P_1, P_2$  are on different rays ( $P_1^{(1)} \neq P_2^{(1)}$ ).

Case (1):  $P_1, P_2$  are on the same ray.

In this case one necessarily has

$$d_2(A, P_1) = d_2(A, P_2) + |P_1^{(2)} - P_2^{(2)}|. \quad (2.7)$$

Case (1.1): Suppose  $A$  is on a different ray as  $P_2$ , then it follows from (2.7) that

$$|A^{(2)}| + |P_1^{(1)} - A^{(1)}| + |P_1^{(2)}| = |A^{(2)}| + |P_2^{(1)} - A^{(1)}| + |P_2^{(2)}| + |P_1^{(2)} - P_2^{(2)}|.$$

Since  $P_1^{(1)} = P_2^{(1)}$ , the above equation is simplified to

$$|P_1^{(2)} - P_2^{(2)}| + |P_2^{(2)}| - |P_1^{(2)}| = 0.$$

This equation holds for all  $A$  with  $A^{(1)} \neq P_2^{(1)}$  provided that  $P_2^{(2)} \in [0, P_1^{(2)})_{\overrightarrow{0 P_1^{(2)}}}$ .

When  $P_2^{(2)} \notin [0, P_1^{(2)})_{\overrightarrow{0 P_1^{(2)}}}$ , it has no solution.

Case (1.2):  $A$  is on the same ray as  $P_2$ . Now one has

$$|A^{(2)} - P_1^{(2)}| = |A^{(2)} - P_2^{(2)}| + |P_1^{(2)} - P_2^{(2)}|.$$

The above equation holds only when

$$A \in \{P_2^{(1)}\} \times [P_2^{(2)}, \infty)_{\overrightarrow{0 P_2^{(2)}}}.$$

Therefore one concludes that when  $P_1$  and  $P_2$  are on the same ray,

$$\mathcal{C}_{d_2}(P_1, P_2) = \begin{cases} \mathbb{R}^2 \setminus (P_2, \infty)_{\overrightarrow{P_1^* P_1}} & \text{if } P_2^{(2)} \in [0, P_1^{(2)})_{\overrightarrow{0 P_1^{(2)}}}; \\ [P_2, \infty)_{\overrightarrow{P_2^* P_2}} & \text{if } P_2^{(2)} \notin [0, P_1^{(2)})_{\overrightarrow{0 P_1^{(2)}}}. \end{cases} \quad (2.8)$$

Case (2):  $P_1, P_2$  are on different rays.

Case (2.1):  $A$  is on the same ray as  $P_1$ . In this case we have

$$\begin{aligned} |A^{(2)} - P_1^{(2)}| &= |A^{(2)}| + |A^{(1)} - P_2^{(1)}| + |P_2^{(2)}| + |P_1^{(2)}| + |P_1^{(1)} - P_2^{(1)}| + |P_2^{(2)}| \\ &> |A^{(2)}| + |P_1^{(2)}|. \end{aligned}$$

This contradicts the triangle inequality, therefore there is no solution for  $A$  in this case.

Case (2.2):  $A$  is on the same ray as  $P_2$ . We have

$$|A^{(2)}| + |A^{(1)} - P_1^{(1)}| + |P_1^{(2)}| = |A^{(2)} - P_2^{(2)}| + |P_1^{(2)}| + |P_1^{(1)} - P_2^{(1)}| + |P_2^{(2)}|.$$

By using the fact that  $A^{(1)} = P_2^{(1)}$ , the above equation becomes

$$|A^{(2)}| = |A^{(2)} - P_2^{(2)}| + |P_2^{(2)}|. \quad (2.9)$$

This provides:

- if  $P_2^{(2)} = 0$ , then the solution space of (2.9) is  $\{P_2^{(1)}\} \times \mathbb{R}$ ;
- if  $P_2^{(2)} \neq 0$ , then the solution space of (2.9) is  $\{P_2^{(1)}\} \times [P_2^{(2)}, \infty)_{0P_2^{(2)}}$ .

Case (2.3):  $A$  is on a different ray as  $P_1, P_2$ .

We have

$$\begin{aligned} |A^{(2)}| + |A^{(1)} - P_1^{(1)}| + |P_1^{(2)}| \\ = |A^{(2)}| + |A^{(1)} - P_2^{(1)}| + |P_2^{(2)}| + |P_1^{(2)}| + |P_1^{(1)} - P_2^{(1)}| + |P_2^{(2)}|. \end{aligned}$$

It is equivalent to

$$|A^{(1)} - P_1^{(1)}| = |A^{(1)} - P_2^{(1)}| + |P_1^{(1)} - P_2^{(1)}| + 2|P_2^{(2)}|.$$

This equation has solution only when  $P_2^{(2)} = 0$ . Then the above equation is written as

$$|A^{(1)} - P_1^{(1)}| = |A^{(1)} - P_2^{(1)}| + |P_1^{(1)} - P_2^{(1)}|.$$

This implies

$$A^{(1)} \in (P_2^{(1)}, \infty)_{P_1^{(1)} P_2^{(1)}}.$$

By combining the solutions for Cases (2.1), (2.2), we obtain, in Case (2),

$$\mathcal{C}_{d_2}(P_1, P_2) = \begin{cases} [P_2, \infty)_{P_2^* P_2} & \text{if } P_1^{(1)} \neq P_2^{(1)}, P_2^{(2)} \neq 0; \\ [P_2^{(1)}, \infty)_{P_1^{(1)} P_2^{(1)}} \times \mathbb{R} & \text{if } P_1^{(1)} \neq P_2^{(1)}, P_2^{(2)} = 0. \end{cases} \quad (2.10)$$

Finally, putting together Cases (1), (2), one proves Proposition 2.3.  $\square$

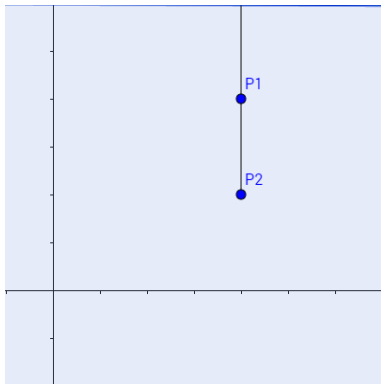


Figure 3: The shaded region represents the set of  $\mathcal{C}_{d_2}(P_1, P_2)$  when  $P_2$  belongs to the segment  $[P_1^*, P_1]$ .

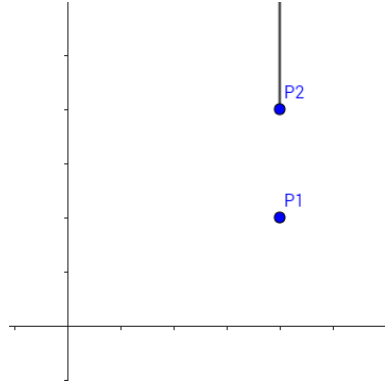


Figure 4: The thick line represents  $\mathcal{C}_{d_2}(P_1, P_2)$  when  $P_1^{(1)} = P_2^{(1)}$  and  $|P_2^{(2)}| > |P_1^{(2)}|$ .



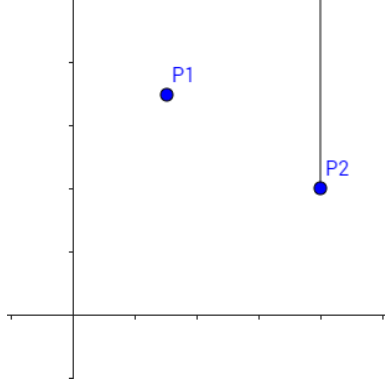


Figure 5: The thick line represents the set of  $\mathcal{C}_{d_2}(P_1, P_2)$  when  $P_1^{(1)} \neq P_2^{(1)}$  and  $P_2^{(2)} \neq 0$ .

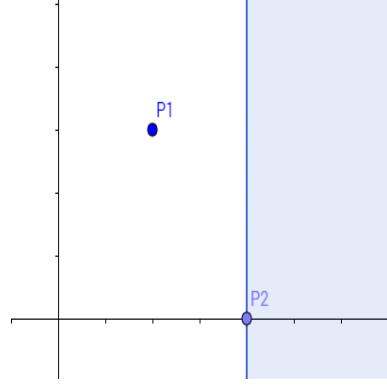


Figure 6: The shaded region represents  $\mathcal{C}_{d_2}(P_1, P_2)$  when  $P_1^{(1)} \neq P_2^{(1)}$  and  $P_2^{(2)} = 0$ .

**Proposition 2.4** *Let  $(\mathbb{R}^n, d)$  be a metric space. If for any  $P_1, P_2 \in \mathbb{R}^n$ , (2.6) holds, then  $(\mathbb{R}^n, d)$  is an  $\mathbb{R}$ -tree.*

**Proof.** We only need to show Condition (A) is satisfied by the expression of  $\mathcal{C}_d(P_1, P_2)$  in (2.6). Observe that for any 3 distinct points  $A, B, C \in \mathbb{R}^n$ , without loss of generality, there are 3 situations according to the positions:

Case 1 :  $A^{(1)} = B^{(1)} = C^{(1)}$ ,  $A^{(2)} \in [0, B^{(2)})_{\overrightarrow{0B^{(2)}}}$ ,  $B^{(2)} \in [0, C^{(2)})_{\overrightarrow{0B^{(2)}}}$ .

Case 2 :  $A^{(1)} = B^{(1)} \neq C^{(1)}$ ,  $A^{(2)} \in [0, B^{(2)})_{\overrightarrow{0B^{(2)}}}$ .

Case 3 :  $A^{(1)}$ ,  $B^{(1)}$  and  $C^{(1)}$  are all distinct,  $B^{(1)} \in [A^{(1)}, C^{(1)}]_{\overrightarrow{A^{(1)}C^{(1)}}}$ .

By (2.6), it is easy to see Condition (A) holds with  $O = B$ ,  $O = A$  and  $O = (0, B^{(2)})$  respectively for Case 1, Case 2 and Case 3. Hence Proposition 2.4 is proven by using Theorem 1.1.  $\square$

### 3 Identification of Radial Metric and River Metric via $\mathcal{C}_d(P_1, P_2)$

#### 3.1 Identification of Radial Metric via $\mathcal{C}_{d_1}(P_1, P_2)$

In Proposition 2.2 and Proposition 2.4, we have shown that the sets  $\{\mathcal{C}_d(P_1, P_2)\}_{P_1, P_2}$  capture the tree properties of the metric spaces  $(\mathbb{R}^n, d_1)$  and  $(\mathbb{R}^2, d_2)$ . Now we claim that subject to some additional conditions these two  $\mathbb{R}$ -trees can be uniquely identified by the sets  $\{\mathcal{C}_d(P_1, P_2)\}_{P_1, P_2}$ .

**Definition 3.1** *Let  $\tilde{d}_1$  be a metric defined on  $\mathbb{R}^n$  satisfying that there exists a function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that*

- *$f$  is continuous;*

- $f$  satisfies the following equation:

$$\begin{cases} \tilde{d}_1(ax, x) = f(|ax - x|) & \text{for all } x \in \mathbb{R}^n \text{ and all } a \geq 0; \\ f(1) = 1. \end{cases}$$

**Theorem 3.1** *The following statements are equivalent:*

- (i)  $\tilde{d}_1 = d_1$ .
- (ii) For any  $P_1, P_2 \in (\mathbb{R}^n, \tilde{d}_1)$ ,  $\mathcal{C}_{\tilde{d}_1}(P_1, P_2) = \mathcal{C}_{d_1}(P_1, P_2)$  given in (2.1).

Before proving Theorem 3.1, we first introduce the following useful statement.

**Theorem 3.2** (See Aczel [1], Theorem 1) *If Cauchy's functional equation*

$$g(u + v) = g(u) + g(v)$$

*is satisfied for all positive  $u, v$ , and if the function  $g$  is*

- *continuous at a point;*
- *nonnegative for small positive  $u - s$  or bounded in an interval,*

*then*

$$g(u) = cu$$

*is the general solution for all positive  $u$ .*

**Proof.** The implication (i)  $\implies$  (ii) is simply Proposition 2.1. Now it remains to prove (ii)  $\implies$  (i).

Case (1):  $A, P_1, 0$  are on the same straight line with  $A \neq P_1$ .

Without loss of generality, assume  $|A| > |P_1|$ . Then there exists  $P_2 \in (P_1, +\infty)_{\overrightarrow{0P_2}}$  such that  $A \in [P_2, +\infty)_{\overrightarrow{0P_2}}$ . By Proposition 2.1, one has

$$\tilde{d}_1(A, P_1) = \tilde{d}_1(A, P_2) + \tilde{d}_1(P_1, P_2).$$

Observe that  $A, P_1, P_2, 0$  are on the same straight line, then by the definition of  $\tilde{d}_1$ , the above equation is equivalent to

$$f(|A - P_1|) = f(|A - P_2|) + f(|P_1 - P_2|). \quad (3.1)$$

This is a Cauchy's equation, then by using Theorem 3.2, the general solution is  $f(u) = cu$ . Together with its initial condition  $f(1) = 1$ , one finally gets

$$f(u) = u.$$

Hence,

$$\tilde{d}_1(A, P_1) = |A - P_1|, \text{ for } A, P_1, 0 \text{ lying on the same straight line.}$$

Case (2):  $A, P_1, 0$  are not on the same straight line (in this case one necessarily has  $A, P_1 \neq 0$ ).

We take  $P_2 = 0$ . The fact that  $A \notin (0, +\infty)_{\overrightarrow{0P_1}}$  implies

$$\tilde{d}_1(A, P_1) = \tilde{d}_1(A, 0) + \tilde{d}_1(P_1, 0).$$

From Case (1) we see

$$\tilde{d}_1(A, 0) = |A| \text{ and } \tilde{d}_1(P_1, 0) = |P_1|.$$

Therefore,

$$\tilde{d}_1(A, P_1) = |A| + |P_1|, \text{ for } A, P_1, 0 \text{ lying on a different straight line.}$$

It follows from Cases (1) and (2) that  $\tilde{d}_1 = d_1$ .  $\square$

### 3.2 Identification of River Metric via $\mathcal{C}_{d_2}(P_1, P_2)$

Now we claim that the inverse statement of Proposition 2.3 holds, under some extra condition.

**Definition 3.2** Define the metric  $\tilde{d}_2$  on  $\mathbb{R}^2$  by

$$\tilde{d}_2(x, y) = \begin{cases} g_1(|x - y|); & \text{if } x_1 = y_1; \\ g_2(|x - y|); & \text{if } x_2 = y_2 = 0, \end{cases}$$

where  $g_1, g_2$  satisfy the same conditions as  $f$  in Definition 3.1.

**Theorem 3.3** The following statements are equivalent:

- (i)  $\tilde{d}_2 = d_2$ .
- (ii) For any  $P_1, P_2 \in (\mathbb{R}^2, \tilde{d}_2)$ ,  $\mathcal{C}_{\tilde{d}_2}(P_1, P_2) = \mathcal{C}_{d_2}(P_1, P_2)$  given in (2.6).

**Proof.** The implication (i)  $\implies$  (ii) is trivial according to Proposition 2.3. Now we prove (ii)  $\implies$  (i).

Case (1):  $A^{(1)} = P_1^{(1)}$ . In this case one takes any  $P_2 \in [P_1, A]_{\overrightarrow{P_1 A}}$  and gets

$$A \in \mathcal{C}_{\tilde{d}_2}(P_1, P_2).$$

Equivalently,

$$\tilde{d}_2(A, P_1) = \tilde{d}_2(A, P_2) + \tilde{d}_2(P_1, P_2).$$

By using the definition of  $g_1$ , one obtains the following Cauchy's equation

$$g_1(|A^{(2)} - P_1^{(2)}|) = g_1(|A^{(2)} - P_2^{(2)}|) + g_1(|P_1^{(2)} - P_2^{(2)}|).$$

Then by Theorem 3.2,

$$g_1(x) = x, \text{ for all } x \geq 0.$$

Case (2) :  $A^{(1)} \neq P_1^{(1)}$ .

Case (2.1): We let  $A^{(2)} = P_1^{(2)} = 0$  and choose  $P_2 = (P_2^{(1)}, 0)$  with  $P_2^{(1)} \in [A^{(1)}, P_1^{(1)}]_{\overrightarrow{P_1^{(1)} A^{(1)}}}$ ,

then by the fact that

$$A \in \mathcal{C}_{\tilde{d}_2}(P_1, P_2),$$

one has

$$\tilde{d}_2(A, P_1) = \tilde{d}_2(A, P_2) + \tilde{d}_2(P_1, P_2).$$

Equivalently,

$$g_2(|A^{(1)} - P_1^{(1)}|) = g_2(|A^{(1)} - P_2^{(1)}|) + g_1(|P_1^{(1)} - P_2^{(1)}|).$$

This Cauchy's equation also implies

$$g_2(x) = x, \text{ for } x \geq 0.$$

Case (2.2):  $A^{(2)} \neq 0, P_1^{(2)} = 0$ . In this case we take  $P_2 = (A^{(1)}, 0)$ , the projection of  $A$  onto the horizontal axis. Therefore by the construction of  $\mathcal{C}_{\tilde{d}_2}(P_1, P_2)$  and

$$\begin{aligned}\tilde{d}_2(A, P_1) &= \tilde{d}_2(A, P_2) + \tilde{d}_2(P_1, P_2) \\ &= |A^{(2)}| + g_2(|P_1 - P_2|) \\ &= |A^{(2)}| + |P_1^{(1)} - A^{(1)}|.\end{aligned}$$

Case (2.3):  $A^{(2)} \neq 0, P_1^{(2)} \neq 0$ . In this case we take  $P_2 = (A^{(1)}, 0)$ , the projection of  $A$  onto the horizontal axis. Therefore by the construction of  $\mathcal{C}_{\tilde{d}_2}(P_1, P_2)$  and

$$\begin{aligned}\tilde{d}_2(A, P_1) &= \tilde{d}_2(A, P_2) + \tilde{d}_2(P_1, P_2) \\ &= |A^{(2)}| + |P_1^{(2)}| + |P_1^{(1)} - P_2^{(1)}| \\ &= |A^{(2)}| + |P_1^{(1)} - A^{(1)}| + |P_1^{(2)}|,\end{aligned}$$

we obtain that in Case (2),

$$\tilde{d}_2(A, P_1) = |A^{(2)}| + |A^{(1)} - P_1^{(1)}| + |P_1^{(2)}|.$$

Finally one obtains for any  $x, y \in \mathbb{R}$ ,

$$\begin{aligned}\tilde{d}_2(x, y) &= \begin{cases} |x_2 - y_2|; & \text{if } x_1 = y_1; \\ |x_2| + |x_1 - y_1| + |y_2|; & \text{if } x_1 \neq y_1 \end{cases} \\ &= d_2(x, y). \quad \square\end{aligned}$$

## 4 An Application: Brownian Motion Indexed by $\mathbb{R}$ -tree

It should be noted that, a tree metric can be also identified by the metric segments  $[A, B]$ , since a uniquely geodesic metric space  $(M, d)$  is a tree if and only if  $[A, B] \cap [B, C] \cap [A, C] = \{O\}$  for all distinct  $A, B, C \in M$ . However, rather than using metric segments, the sets  $\mathcal{C}(P_1, P_2)$  allow to capture the features of a Gaussian field, which has very important and interesting applications in the domain of random fields. As an example, Inoue and Nota (1982) [7] studied some classes of Gaussian fields on  $(\mathbb{R}^n, |\cdot|)$  and represented them via the sets of independent increments. Namely, some random field  $\{X(t)\}_{t \in \mathbb{R}^n}$  can be identified by the sets: for any  $P_1, P_2 \in \mathbb{R}^n$ ,

$$\mathcal{F}_X(P_1|P_2) = \{A \in \mathbb{R}^n : \text{Cov}(X(A) - X(P_2), X(P_1) - X(P_2)) = 0\}.$$

The set  $\mathcal{F}_X(P_1|P_2)$  satisfies the property that, the increments  $X(A) - X(B)$  and  $X(P_1) - X(P_2)$  are mutually independent if and only if  $A, B \in \mathcal{F}_X(P_1|P_2)$ . Here, we take a very similar idea of representation Gaussian fields, but work with a tree metric which is different from Euclidean distance  $|\cdot|$ . More precisely, we notice that a mean zero Brownian motion  $B$  indexed by an  $\mathbb{R}$ -tree  $(M, d)$  is well-defined, from its initial value  $B(O) = 0$  and its covariance structure

$$\text{Cov}(B(X), B(Y)) = \frac{1}{2} (d(O, X) + d(O, Y) - d(X, Y)). \quad (4.1)$$

In fact, since a tree metric  $d$  is of negative type, the mapping

$$(X, Y) \mapsto \frac{1}{2} (d(O, X) + d(O, Y) - d(X, Y))$$

is thus positive definite so it well defines a Brownian motion covariance function.

Let  $\mathcal{C}_d(P_1, P_2)$  be the one corresponding to  $(M, d)$ . Then by a similar study in [7], we see that, not only  $\mathcal{C}_d(P_1, P_2)$  can be used to identify the Bm  $B$ , but for any  $X, Y \in \mathcal{C}_d(P_1, P_2)$ , one has  $B(X) - B(Y)$  and  $B(P_1) - B(P_2)$  are independent. This is due to the fact that, by using (4.1) and the definition of  $\mathcal{C}_d(P_1, P_2)$ ,

$$\mathcal{F}_B(P_1|P_2) = \mathcal{C}_d(P_1, P_2), \text{ for any } P_1, P_2 \in \mathbb{R}^n.$$

Hence

$$X, Y \in \mathcal{C}_d(P_1, P_2)$$

implies

$$\text{Cov}(B(X) - B(Y), B(P_1) - B(P_2)) = 0.$$

As a consequence  $\{\mathcal{C}_d(P_1, P_2)\}_{P_1, P_2 \in M}$  captures all sets of independent increments of  $\{B(X)\}_{X \in (M, d)}$ . By this way one creates a new strategy to detect and simulate Brownian motion indexed by an  $\mathbb{R}$ -tree.

## 4.1 Identification of Brownian Motions Indexed by $\mathbb{R}$ -trees

Let  $\{B(X)\}_{X \in (\mathbb{R}^n, d)}$  be a zero mean Brownian motion indexed by an  $\mathbb{R}$ -tree. Namely,  $\mathbb{E}(B(X)) = 0$  for all  $X \in \mathbb{R}^n$  and there exists an initial point  $O$  such that (4.1) holds. Then the theorems below easily follow from Theorem 3.1 and Theorem 3.3 respectively.

**Theorem 4.1** *The following statements are equivalent:*

- (i)  $d = d_1$ .
- (ii) For any  $P_1, P_2 \in \mathbb{R}^n$ ,  $\mathcal{F}_B(P_1|P_2) = \mathcal{C}_{d_1}(P_1, P_2)$ .

**Theorem 4.2** *For Brownian motion indexed by  $(\mathbb{R}^2, d_2)$ , the following statements are equivalent:*

- (i)  $d = d_2$ .
- (ii) For any  $P_1, P_2 \in \mathbb{R}^2$ ,  $\mathcal{F}_B(P_1|P_2) = \mathcal{C}_{d_2}(P_1, P_2)$ .

## 4.2 Simulation of Brownian Motion Indexed by $\mathbb{R}$ -tree

Let us consider a Brownian motion  $B$  indexed by a tree  $(\mathbb{R}^2, d_1)$  (recall that  $d_1$  denotes radial metric) as an example. An interesting question in statistics is to simulate such a Brownian motion. More precisely, the issue is how can we generate the sample path  $\{B(A_1), \dots, B(A_n)\}$ , for any different  $A_1, \dots, A_n \in (\mathbb{R}^2, d_1)$ ? In this section, we propose a new approach, which relies on the set  $\mathcal{F}_B(P_1|P_2)$ .

The following proposition shows, in some special case, the simulation could be particularly simple.

**Proposition 4.1** *For any  $A_1, \dots, A_n \in (\mathbb{R}^2, d_1)$ , there exists a permutation  $\sigma \in S_n$  ( $S_n$  denotes the group of permutations of  $\{1, 2, \dots, n\}$ ) and an integer  $q \geq 1$  with  $n_1 + n_2 + \dots + n_q = n$ , such that*

$$\begin{aligned} & (B(A_{\sigma(1)}), \dots, B(A_{\sigma(n_1)})), (B(A_{\sigma(n_1+1)}), \dots, B(A_{\sigma(n_1+n_2)})), \\ & \dots, (B(A_{\sigma(n_1+\dots+n_{q-1}+1)}), \dots, B(A_{\sigma(n)})) \end{aligned} \quad (4.2)$$

are independent, and for each group, i.e., for  $1 \leq l \leq q$ ,

$$(B(A_{\sigma(n_1+\dots+n_{l-1}+1)}), \dots, B(A_{\sigma(l)})) \quad (4.3)$$

has independent increments.

**Proof.** It suffices to provide a such  $\sigma$ . We first transform  $A_1, \dots, A_n$  to their polar coordinates representations. For each  $A_k$  where  $k \in \{1, \dots, n\}$ , there exists  $r_k \in [0, +\infty)$  and  $\theta_k \in [0, 2\pi)$  such that  $A_k = r_k e^{i\theta_k}$ . The following approach provides a permutation  $\sigma$  satisfying (4.2): we choose  $\sigma \in S_n$  such that

$$\theta_{\sigma(1)} = \dots = \theta_{\sigma(n_1)} < \theta_{\sigma(n_1+1)} = \dots = \theta_{\sigma(n_1+n_2)} < \dots < \theta_{\sigma(n_1+\dots+n_{q-1}+1)} = \dots = \theta_{\sigma(n)}$$

with  $n_1 + \dots + n_q = n$  and for each group  $\sigma(\sum_{m=1}^l n_m + 1), \dots, \sigma(\sum_{m=1}^{l+1} n_m)$ ,

$$r_{\sigma(\sum_{m=1}^l n_m + 1)} \leq r_{\sigma(\sum_{m=1}^l n_m + 2)} \leq \dots \leq r_{\sigma(\sum_{m=1}^{l+1} n_m)}.$$

To show (4.2) and (4.3), on one hand, by Theorem 4.1, for each  $l = 1, \dots, n$ , the elements  $\{A_k\}_{k=\sigma(\sum_{m=1}^l n_m + 1), \dots, \sigma(\sum_{m=1}^{l+1} n_m)}$  are on the same ray so they have independent increments. On the other hand, the random vectors

$$(B(A_{\sigma(1)}), \dots, B(A_{\sigma(n_1)})), (B(A_{\sigma(n_1+1)}), \dots, B(A_{\sigma(n_1+n_2)})), \\ (B(A_{\sigma(n_1+\dots+n_{q-1}+1)}), \dots, B(A_{\sigma(n)}))$$

are independent, due to the fact that for  $X, Y$  on different rays,

$$\begin{aligned} \text{Cov}(B(X), B(Y)) &= \frac{1}{2} (d_1(X, 0) + d_1(Y, 0) - d_1(X, Y)) \\ &= \frac{1}{2} (|X| + |Y| - (|X| + |Y|)) = 0. \quad \square \end{aligned}$$

Proposition 4.1 leads to the following simulation algorithm.

#### 4.2.1 Algorithm of Simulating Brownian Motion Indexed by $(\mathbb{R}^2, d_1)$ :

If  $A_1, \dots, A_n$  verify the assumption given in Proposition 4.1, then

Step 1: Determine  $\sigma \in S_n$  and  $q \geq 1$  such that

$$(B(A_{\sigma(1)}), \dots, B(A_{\sigma(n_1)})), (B(A_{\sigma(n_1+1)}), \dots, B(A_{\sigma(n_1+n_2)})), \\ \dots, (B(A_{\sigma(n_1+\dots+n_{q-1}+1)}), \dots, B(A_{\sigma(n)}))$$

are independent, and each vector has independent increments.

Step 2: Generate  $n$  independent zero mean Gaussian random variables  $Z_1, \dots, Z_n$ , with

$$\text{Var}(Z_k) = \begin{cases} d_1(0, A_{\sigma(k)}) & \text{if } k = \sum_{m=1}^l n_m + 1 \text{ for some } m \\ d_1(A_{\sigma(k-1)}, A_{\sigma(k)}) & \text{otherwise.} \end{cases}$$

Step 3: For  $j = 1, \dots, n$ , set

$$B(A_{\sigma(j)}) = \sum_{k=\sum_{m=1}^l n_m + 1}^j Z_k, \text{ if } j \in \left\{ \sum_{m=1}^l n_m + 1, \dots, \sum_{m=1}^{l+1} n_m \right\}.$$

Now let us study the simulation of Brownian motion  $B$  indexed by  $(\mathbb{R}^2, d_2)$ , an  $\mathbb{R}$ -tree with river metric. Similar to Proposition 4.1, we have the following proposition:

**Proposition 4.2** *Given  $n$  points vertically and horizontally labelled, i.e., for  $A_1, \dots, A_n \in (\mathbb{R}^2, d_2)$  such that*

$$\{(0, 0), (A_1^{(1)}, 0), \dots, (A_n^{(1)}, 0)\} \subset \{A_1, \dots, A_n\}$$

and

$$\begin{aligned} A_1^{(1)} &= \dots = A_{n_1}^{(1)} < A_{n_1+1}^{(1)} = \dots = A_{n_1+n_2}^{(1)} \\ &< \dots < A_{n_1+\dots+n_{p-1}+1}^{(1)} = \dots = A_{n_1+\dots+n_p}^{(1)} < 0 \\ &\leq \dots < A_{n_1+\dots+n_{q-1}+1}^{(1)} = \dots = A_n^{(1)} \end{aligned}$$

with  $n_1 + \dots + n_q = n$  and for each group  $\sum_{m=1}^l n_m + 1, \dots, \sum_{m=1}^{l+1} n_m$ ,

$$A_{\sum_{m=1}^l n_m+1}^{(2)} \leq A_{\sum_{m=1}^l n_m+2}^{(2)} \leq \dots \leq A_{\sum_{m=1}^{l+1} n_m}^{(2)}.$$

Then there exists a sequence of independent Gaussian variables  $(Z_1, \dots, Z_{n-1})$  such that

$$(B(A_1), \dots, B(A_n)) = \left( \sum_{k \in I_1} Z_k, \dots, \sum_{k \in I_n} Z_k \right) \text{ in distribution} \quad (4.4)$$

for some  $I_k \subset \{1, \dots, n\}$  for any  $k = 1, \dots, n$ .

**Proof.** We define for  $k = 1, \dots, n-1$ ,

$$Z_k = \begin{cases} B(A_{k+1}) - B(A_k) & \text{if } A_{k+1}^{(1)} = A_k^{(1)} \\ B((A_{k+1}^{(1)}, 0)) - B((A_k^{(1)}, 0)) & \text{if } A_{k+1}^{(1)} > A_k^{(1)} \end{cases}. \quad (4.5)$$

From Theorem 4.2, we see  $(Z_1, \dots, Z_{n-1})$  is a sequence of independent random variables. Now we are going to find  $I_1, \dots, I_n$  such that (4.4) holds true. Let's consider a directed graph  $G = (V, E)$ , with the set of vertices

$$V = \{A_1, \dots, A_n\}$$

and the set of edges

$$E = \{e_1, \dots, e_{n-1}\},$$

where

$$e_k = \begin{cases} \overrightarrow{A_k A_{k+1}} & \text{if } A_{k+1}^{(1)} = A_k^{(1)} \geq 0 \\ \overrightarrow{(A_k^{(1)}, 0)(A_{k+1}^{(1)}, 0)} & \text{if } A_{k+1}^{(1)} > A_k^{(1)} \geq 0 \\ \overrightarrow{A_k A_{k-1}} & \text{if } A_{k-1}^{(1)} = A_k^{(1)} < 0 \\ \overrightarrow{(A_k^{(1)}, 0)(A_{k-1}^{(1)}, 0)} & \text{if } A_{k-1}^{(1)} < A_k^{(1)} < 0. \end{cases} \quad (4.6)$$

We denote by  $A_{n_0} = (0, 0)$ . For  $k = 1, \dots, n$ , let  $P_k$  be the shortest path from  $A_{n_0}$  to  $A_k$  in  $G$ . Namely, there exists a set  $\{k_1, \dots, k_{\psi(k)}\} \subset \{1, \dots, n\}$  such that

$$\begin{aligned} P_k &= \left( \overrightarrow{A_{n_0} A_{k_1}}, \overrightarrow{A_{k_1} A_{k_2}}, \dots, \overrightarrow{A_{k_{\psi(k)-1}} A_{k_{\psi(k)}}} \right) \\ &= (e_{j_1}, \dots, e_{j_{\psi(k)}}). \end{aligned}$$

Denote by  $I_k = \{j_1, \dots, j_{\psi(k)}\}$ , then (4.4) is satisfied for such a choice of  $(I_k)_{k=1, \dots, n}$ .

□

Proposition 4.2 leads to the following simulation algorithm.

#### 4.2.2 Algorithm of Simulating Brownian Motion Indexed by $(\mathbb{R}^2, d_2)$ :

If  $A_1, \dots, A_n \in \mathbb{R}^2$ , the following algorithm shows how to simulate  $(B(A_1), \dots, B(A_n))$ :

Step 1:

Generate  $n-1$  independent zero mean Gaussian random variables  $Z_1, \dots, Z_{n-1}$ , with

$$Var(Z_k) = \begin{cases} d_2(A_k, A_{k+1}) & \text{if } A_{k+1}^{(1)} = A_k^{(1)} \\ d_2((A_{k+1}^{(1)}, 0), (A_k^{(1)}, 0)) & \text{if } A_{k+1}^{(1)} > A_k^{(1)}. \end{cases}$$

Step 2: For  $k = 1, \dots, n$ , determine  $I_k$ . Finally,

$$(B(A_k))_{k=1, \dots, n} = \left( \sum_{j \in I_k} Z_j \right)_{k=1, \dots, n} \quad \text{in distribution.}$$

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